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Lagrange versus symplectic algorithm for constrained systems

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Abstract

The systematization of the purely Lagrangian approach to constrained systems in the form of an algorithm involves the iterative construction of a generalized Hessian matrix W taking a rectangular form. This Hessian will exhibit as many left zero modes as there are Lagrangian constraints in the theory. We apply this approach to a general Lagrangian in the first-order formulation and show how the seemingly overdetermined set of equations is solved for the velocities by suitably extending W to a rectangular matrix. As a byproduct we thereby demonstrate the equivalence of the Lagrangian approach to the traditional Dirac approach. By making use of this equivalence we show that a recently proposed symplectic algorithm does not necessarily reproduce the full constraint structure of the traditional Dirac algorithm.

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1. Introduction

A number of algorithms have been developed over the past few years for treating constrained Hamiltonian systems. Perhaps the most familiar one to the physicist community is the one developed by Dirac [1]. Although very elegant and powerful in its algebraic structure, this algorithm has been criticized for being based on the existence of so-called 'primary constraints', which are a purely phase-space artefact, and have no counterpart on the Lagrangian level. Faddeev and Jackiw [2] have thus proposed an alternative approach based on a first-order Lagrangian formulation, avoiding the introduction of primary constraints. Furthermore, the local symmetries of the Hamiltonian, as generated by the so-called 'first-class' constraints in Dirac's terminology, turn out to be larger than those of the Lagrangian [3]. This has led to a renewed interest in deducing directly the local symmetries of a Lagrangian within a purely Lagrangian approach in the form of an algorithm [4]. Parallelling this approach a 'symplectic algorithm' has been proposed in [5] for constrained systems, based on a first-order Lagrangian formulation. This method was elaborated in [6, 7], and has been extensively applied to a variety of models [8–11].

Of all three methods, the Lagrangian algorithm is actually the most pedestrian one. The constraints generated by this algorithm are identical with those obtained in the Dirac approach. On the other hand, the algorithm proposed for first-order Lagrangians in [5] lacks a deductive mathematical justification, and can lead, as we shall demonstrate, to an incomplete solution of the problem. In order to establish the relation between the three formalisms, we shall thus take as our starting point the Lagrangian approach as applied to first-order Lagrangians, in order to allow for a comparison with the approach of [5]. As shown in section 2, the Lagrangian algorithm leads to a larger set of equations than the number of unknown velocities to be solved for. This is reflected in the fact that the generalized Hessian which implements the algorithm is a *rectangular* matrix possessing as many left zero modes as there are Lagrangian constraints hidden in the Euler–Lagrange equations. We show that these zero modes are of such a form that they permit the solution of the equations of motion in terms of the inverse of a quadratic matrix, whose elements are just the Poisson brackets of the Hamiltonian constraints—including the primary constraints. We thereby establish the equivalence with Dirac's algorithm. In section 3, we then consider the simple example of particle motion on a hypersphere and thereby demonstrate that the symplectic algorithm of [5] is not always equivalent to the Dirac and Lagrangian approach. We conclude this section by discussing the general condition under which this symplectic algorithm fails. Section 5 summarizes our findings.

2. The Lagrangian algorithm

Given a second-order Lagrangian, one can always find an equivalent first-order Lagrangian of the form

$$L(Q, \dot{Q}) = a_\alpha(Q)\dot{Q}_\alpha - V(Q) \tag{1}$$

where Q stands for n degrees of freedom $Q_\alpha, \alpha = 1, 2, \ldots, n$. The corresponding Euler–Lagrange equations read

$$W^{(0)}_{\alpha\beta}(Q)\dot{Q}_\beta = \frac{\partial V(Q)}{\partial Q_\alpha} \tag{2}$$

where the matrix $W^{(0)}$ is defined by

$$W^{(0)}_{\alpha\beta}(Q) = \frac{\partial a_\beta}{\partial Q_\alpha} - \frac{\partial a_\alpha}{\partial Q_\beta}. \tag{3}$$

Let r_0 be the rank of the matrix $W^{(0)}$. Then there exist $n - r_0$ zero modes of $W^{(0)}$, which we denote by $u^{(0)}(a), a = 1, \ldots, n - r_0$. Multiplying equations (2) from the left with these zero modes, we are led to the zero-level Lagrangian constraints

$$\varphi^{(0)}_a = \sum_\alpha u^{(0)}_\alpha(a)\frac{\partial V}{\partial Q_\alpha} = 0 \qquad a = 1, \ldots, n_0. \tag{4}$$

Some of these constraints may vanish identically. The remaining ones we denote by $\varphi^{(0)}_{a_0}$. The corresponding zero modes $u^{(0)}(a_0)$ we refer to as 'non-trivial'.

In general there are further constraints hidden in equations (2). In order to unravel them, we implement their conservation by adjoining their time derivatives

$$\left(\frac{\partial \varphi^{(0)}_{a_0}}{\partial Q_\alpha}\right)\dot{Q}_\alpha = 0 \tag{5}$$

to equations (2). This leads to the following enlarged set of equations

$$W^{(1)}_{A_1\beta}(Q)\dot{Q}_\beta = K^{(1)}_{A_1}(Q) \tag{6}$$

where $W^{(1)}_{A_1\beta}$ are now the elements of a rectangular matrix

$$W^{(1)}_{A_1\beta} := \begin{pmatrix} W^{(0)}_{\alpha\beta} \\ M^{(0)}_{a_0\beta} \end{pmatrix} \tag{7}$$

with

$$M^{(0)}_{a_0\beta} = \frac{\partial\varphi^{(0)}_{a_0}}{\partial Q_\beta} \qquad K^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \vec{0} \end{pmatrix} \tag{8}$$

where

$$\vec{K}^{(0)} = \vec{\nabla}V(Q). \tag{9}$$

We now look for 'non-trivial' zero modes $(u^{(1)}(a_1), a_1 = 1, \ldots, n_1)$ of $W^{(1)}$, and repeat the steps above, adjoining the time derivative of any new constraints to the equations of motion (6). Repeating this algorithm, the iterative process terminates after L steps, when no new constraints are generated.

We denote the full set of constraints generated by the algorithm collectively by $\{\varphi_a\}, a = 1, \ldots, N$. We denote further the set $\{\alpha, a\}$ collectively by $\{A\}$. The *final* set of equations can then be written in the form

$$W_{A\beta}\dot{Q}_\beta = K_A \tag{10}$$

where

$$W_{A\beta} := \begin{pmatrix} W^{(0)}_{\alpha\beta} \\ M_{a\beta} \end{pmatrix} \tag{11}$$

with

$$M_{a\beta} = \frac{\partial\varphi_a}{\partial Q_\beta} \qquad K_A := \begin{pmatrix} \vec{K}^{(0)} \\ \vec{0} \end{pmatrix}. \tag{12}$$

Denoting by $\vec{u}(a)$ the left zero modes of the matrix $W_{A\beta}$, the constraints are given by $\varphi_a = \vec{u}(a) \cdot \vec{K} = 0$.

Equations (10) represent $n + N$ equations for the n velocities $\{\dot{Q}_\alpha\}$. In general, such a set of equations would be overdetermined and admit no non-trivial solution. Since the additional N equations were however generated by a self-consistent algorithm from the original Euler–Lagrange equations, equations (10) do in fact admit a non-trivial solution. In the following, we shall assume the first-order Lagrangian (1) to describe a purely second-class system in the Dirac terminology. In that case we have the following:

Assertion. *The unique solution to (10) for the velocities is given by*

$$\dot{Q}_\alpha = F^{-1}_{\alpha\beta}K^{(0)}_\beta \tag{13}$$

where F^{-1} is the inverse of the matrix F obtained by extending the rectangular matrix W defined in (11) to the antisymmetric square matrix

$$F_{AB} := \begin{pmatrix} W^{(0)}_{\alpha\beta} & -M^T_{\alpha b} \\ M_{a\beta} & \mathbf{0} \end{pmatrix} \tag{14}$$

with $M_{a\beta}$ defined in (12), and $M^T_{\alpha b} = M_{b\alpha}$.

We now prove this assertion. Consider an enlarged space on which the square matrix (14) is to act (we streamline the notation in a self-evident way),

$$\xi_A := (Q_\alpha, \rho_a) \tag{15}$$

and the following equations:

$$F_{AB}\dot{\xi}_B = K_A. \tag{16}$$

As we shall prove further below, det $F \neq 0$ for a second-class system. Hence we can solve these equations for the velocities $\dot{\xi}_B$:

$$\dot{\xi}_A = F^{-1}_{AB} K_B. \tag{17}$$

We write the inverse matrix F^{-1} in the form

$$F^{-1}_{AB} := \begin{pmatrix} \tilde{W}^{(0)}_{\alpha\beta} & -\tilde{M}^T_{\alpha b} \\ \tilde{M}_{a\beta} & \omega_{ab} \end{pmatrix}. \tag{18}$$

Then $F^{-1}F = 1$ and $FF^{-1} = 1$ respectively imply in particular,

$$\tilde{M}_{a\gamma} W^{(0)}_{\gamma\beta} + \omega_{ac} M_{c\beta} = 0 \tag{19}$$

$$\tilde{M}_{a\gamma} M^T_{\gamma b} = -\delta_{ab}. \tag{20}$$

Consider equations (16) which, written out explicitly, read

$$W^{(0)}_{\alpha\beta}\dot{Q}_\beta - M^T_{\alpha b}\dot{\rho}_b = K^{(0)}_\alpha \tag{21}$$

$$M_{a\beta}\dot{Q}_\beta = 0. \tag{22}$$

From (12) we see that the last equation states that $\dot{\varphi}_a = 0$, where $\varphi_a = 0$ are the constraints hidden in the equations of motion (2). Because of this, requiring their persistence in time implies that the second term on the LHS of (21) must vanish. Making use of (20) this in turn implies that $\dot{\rho}_a = 0$ for all a.[1] Setting $\dot{\rho}_a = 0$ in (17), we have from (18),

$$\dot{Q}_\alpha = \tilde{W}^{(0)}_{\alpha\beta} K^{(0)}_\beta \tag{23}$$

$$0 = \tilde{M}_{a\beta} K^{(0)}_\beta. \tag{24}$$

Equation (24) is just the statement that $\varphi_a = 0$. To see this we note that according to (19), the vectors

$$\vec{u}_A(a) := (\tilde{M}_{a\gamma}, \omega_{ac}) \tag{25}$$

are just the left zero modes of the matrix (11). Hence

$$\tilde{M}_{a\beta} K^{(0)}_\beta = u_A(a) K_A = \varphi_a. \tag{26}$$

As we now show, equations (17) are nothing other than the Hamilton equations of motion derived from the so-called extended Hamiltonian. By making contact with the Hamiltonian formalism, we will prove that (i) F is an invertible matrix, and (ii) the solutions to (21) imply that $\dot{\rho}_a = 0$, as was claimed above. This, at the same time, will prove the uniqueness of the solution.

From the Dirac point of view, the symplectic Lagrangian (1) describes a system with a *primary* constraint for every coordinate Q_α:

$$\phi_\alpha := P_\alpha - a_\alpha(Q) = 0 \qquad \alpha = 1, \ldots, n \tag{27}$$

[1] An alternative proof that this must indeed be the case will be given further below, where we make contact with the Hamiltonian formalism.

where P_α are the momenta canonically conjugate to the coordinates Q_α. Since the Lagrangian (1) is first-order in the time derivatives, the corresponding canonical Hamiltonian H_c is just given by the potential V,

$$H_c = V(Q) \tag{28}$$

and hence does not depend on the momenta. The dependence on the momenta enters only in the *total* Hamiltonian via the primary constraints:

$$H_T(Q, P) = V(Q) + \sum_\alpha v_\alpha \phi_\alpha(Q, P). \tag{29}$$

The Dirac algorithm will in general lead to secondary constraints, which we label by a latin index: $\varphi_a = 0$. It is easy to see that they are identical with the constraints generated by the Lagrangian algorithm. Thus consider the persistence equations for the primary constraints ϕ_α:

$$\{\phi_\alpha, H_T\} = \{\phi_\alpha, V\} + \sum_\beta \{\phi_\alpha, \phi_\beta\} v_\beta = 0.$$

From (27) and (3) we see that $\{\phi_\alpha, \phi_\beta\} = W^{(0)}_{\alpha\beta}(Q)$, so that the above equations read

$$W^{(0)}_{\alpha\beta} v_\beta = K^{(0)}_\alpha. \tag{30}$$

Multiplying this equation with the left-zero modes of $W^{(0)}$ we arrive at the level-zero Lagrangian constraints (4), which are only functions of Q. Requiring their persistence in time as generated by H_T yields $M^{(0)}_{a\beta} v_\beta = 0$, and adjoining these equations to (30),

$$W^{(1)}_{A_1\beta} v_\beta = K^{(1)}_{A_1}.$$

By taking appropriate linear combinations of these equations, new constraints may be generated which are functions of only the Q_α. This just corresponds to looking for left-zero modes of $W^{(1)}$. The new constraints are thus identical with those derived in the Lagrangian approach at level 'one'. Proceeding in this way it is easy to see that the secondary constraints generated by the Dirac algorithm applied to $H_T(Q, P)$ are identical with the constraints $\{\varphi_a = 0\}$ generated by the Lagrangian algorithm.

We now go over to the extended Hamiltonian by including the secondary constraints with their respective Lagrange multipliers $\bar{v}_a$,

$$H_T \to H_E = H_c + \sum_B \lambda_B \Omega_B \tag{31}$$

where

$$\Omega_A := (\phi_\alpha, \varphi_a) \qquad \lambda_A := (v_\alpha, \bar{v}_a).$$

The Hamilton equations of motion for the coordinates Q_α associated with the extended Hamiltonian H_E, read

$$\dot{Q}_\alpha = \{Q_\alpha, H_E\} = v_\alpha \tag{32}$$

$$\dot{P}_\alpha = \{P_\alpha, H_E\} = -K^{(0)}_\alpha + v_\beta \partial_\alpha a_\beta - \bar{v}_b \partial_\alpha \varphi_b \tag{33}$$

where a_α has been defined in (1). Consistency with the persistence in time of the primary constraints requires

$$\dot{\phi}_\alpha = \dot{P}_\alpha - \dot{a}_\alpha = 0. \tag{34}$$

One readily verifies from (33) that this leads to

$$W^{(0)}_{\alpha\beta} v_\beta - M^T_{\alpha b} \bar{v}_b = K^{(0)}_\alpha. \tag{35}$$

On the other hand, persistence of the secondary constraints φ_a leads to

$$M_{b\beta}v_\beta = 0. \tag{36}$$

Upon making use of (32), we thus retrieved equations (21) and (22) if we identify $\bar{v}_a$ with $\dot{\rho}_a$. Hence in the Hamiltonian formalism these equations are merely the persistence equations of the primary and secondary constraints, which can be compactly written in the Hamiltonian form

$$\{\Omega_A, H_c\} + \sum_B \{\Omega_A, \Omega_B\}\lambda_B = 0. \tag{37}$$

We now recognize that the matrix elements of F in (14) are given by

$$F_{AB} = \{\Omega_A, \Omega_B\}.$$

Since the constraints have been assumed to be second class, this matrix is invertible. Noting further that

$$\{\Omega_A, H_c\} = -K_A$$

it follows from (37) that

$$\lambda_A = F^{-1}_{AB}K_B. \tag{38}$$

With $\dot{Q}_\alpha = v_\alpha$, these equations are nothing other than (17), with $\dot{\rho}_a$ identified with $\bar{v}_a$.

To prove the equivalence of (35) and (36) with the original set of equations (10), we must still show that equations (38) imply that $\bar{v}_a = 0$. To this effect we recall that the secondary constraints $\varphi_a = 0$ have actually been generated by the *total* Hamiltonian H_T from the persistence equations

$$\{\Omega_A, H_c\} + \sum_\beta \{\Omega_A, \phi_\beta\}\lambda_\beta = 0.$$

Consistency with (37) therefore requires that

$$\sum_b \{\Omega_A, \varphi_b\}\bar{v}_b = 0$$

or equivalently $M^T_{\alpha b}\bar{v}_b = 0$, which, upon making use of (20), implies $\bar{v}_a = 0$. This completes the proof of our assertion.

Concluding this section we have therefore shown the full equivalence of the Lagrangian and Hamiltonian approach to the theory described by the first-order Lagrangian (1). Any other approach must therefore reproduce the constrained structure of the Lagrangian approach. An alternative algorithm for unravelling the constrained structure was proposed in [5]. We shall refer to it as the BW symplectic algorithm. In the following section, we will show that this algorithm does not necessarily generate the correct constrained structure.

3. The BW symplectic algorithm

In the following we first illustrate in terms of a simple example, an alternative algorithm for generating the constraints, as proposed in [5].

3.1. Particle on a hypersphere

The following (second-order) Lagrangian is referred to as describing the non-linear sigma model in quantum mechanics:

$$L = \tfrac{1}{2}\dot{\vec{q}}^2 + \lambda(\vec{q}^2 - 1) \tag{39}$$

where $\vec{q} = (q_1, \ldots, q_n)$. The equivalent first-order Lagrangian reads [5]

$$L^{(0)} = a_\alpha(Q)\dot{Q}_\alpha - V^{(0)}(Q) \tag{40}$$

with

$$Q_\alpha := (\vec{q}, \vec{p}, \lambda) \qquad a_\alpha := (\vec{p}, \vec{0}, 0) \tag{41}$$

and

$$V^{(0)} = -\lambda(\vec{q}^2 - 1) + \tfrac{1}{2}\vec{p}^2. \tag{42}$$

The equations of motion are of the form (2), with

$$W^{(0)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \end{pmatrix} \qquad K^{(0)}_\alpha = \begin{pmatrix} -2\lambda\vec{q} \\ \vec{p} \\ -(\vec{q}^2 - 1) \end{pmatrix}. \tag{43}$$

The matrix $W^{(0)}$ has one 'zero-level' zero mode:

$$u^{(0)}_\alpha := (\vec{0}, \vec{0}, 1)$$

implying the constraint

$$\varphi^{(0)} = -u^{(0)}_\alpha K^{(0)}_\alpha = \vec{q}^2 - 1 = 0. \tag{44}$$

This constraint will necessarily coincide with that of the Lagrangian approach at the zeroth level.

In the BW symplectic algorithm the time derivative of the constraint (44) is however added in the (partially integrated) form $-\dot{\rho}^{(0)}\varphi^{(0)}$ to the zero-level Lagrangian (40),[2] to yield the first-level Lagrangian

$$L^{(1)} = L^{(0)} - \dot{\rho}^{(0)}\varphi^{(0)} \tag{45}$$

where $\rho^{(0)}$ is a new dynamical variable. Correspondingly, we define the extended set of coordinates

$$\xi^{(1)}_{A_1} := (\vec{q}, \vec{p}, \lambda, \rho^{(0)}).$$

$L^{(1)}$ can be written in the form

$$L^{(1)} = a^{(1)}_{A_1}\dot{\xi}^{(1)}_{A_1} - V^{(0)}(Q)$$

where

$$a^{(1)}_{A_1} := (\vec{p}, \vec{0}, 0, -\varphi^{(0)}).$$

The corresponding Euler–Lagrange equations read

$$F^{(1)}_{A_1B_1}\dot{\xi}^{(1)}_{B_1} = K^{(1)}_{A_1}$$

where the 'first-level' symplectic *square* matrix $F^{(1)}$, and $K^{(1)}$ are given by

$$F^{(1)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} & -2\vec{q} \\ \mathbf{1} & \mathbf{0} & \vec{0} & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 & 0 \\ 2\vec{q}^T & \vec{0}^T & 0 & 0 \end{pmatrix} \qquad K^{(1)} = \begin{pmatrix} -2\lambda\vec{q} \\ \vec{p} \\ -(\vec{q}^2 - 1) \\ 0 \end{pmatrix}. \tag{46}$$

[2] In [5], the term proportional to λ in $L^{(0)}$ has been absorbed into the term proportional to $\dot{\rho}$.

$F^{(1)}$ has two (level-one) zero modes,

$$u^{(1)}_{A_1}(1) := (\vec{0}, \vec{0}, 1, 0) \qquad u^{(1)}_{A_1}(2) := (\vec{0}, 2\vec{q}, 0, -1).$$

The first zero mode reproduces the constraint $\vec{q}^2 - 1 = 0$. The second zero mode yields the new constraint

$$\varphi^{(1)} = \vec{u}^{(1)}(2) \cdot \vec{K}^{(1)} = 2\vec{p} \cdot \vec{q}.$$

As one easily verifies, these constraints are identical with those obtained in the Lagrangian algorithm described in section 1, at this level.

According to the BW symplectic algorithm we now define the second-level Lagrangian by adding the new constraint in the form

$$L^{(2)} = L^{(0)} - \dot{\rho}^{(0)}\varphi^{(0)} - \dot{\rho}^{(1)}\varphi^{(1)}$$

or

$$L^{(2)} = a^{(2)}_{A_2}\dot{\xi}^{(2)}_{A_2} - V^{(0)}(Q)$$

with

$$\xi^{(2)}_{A_2} := (\vec{q}, \vec{p}, \lambda, \rho^{(0)}, \rho^{(1)})$$

and

$$a^{(2)}_{A_2} := (\vec{p}, \vec{0}, 0, -(\vec{q}^2 - 1), -2\vec{p} \cdot \vec{q}).$$

For the corresponding symplectic matrix one obtains

$$F^{(2)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} & -2\vec{q} & -2\vec{p} \\ \mathbf{1} & \mathbf{0} & \vec{0} & \vec{0} & -2\vec{q} \\ \vec{0}^T & \vec{0}^T & 0 & 0 & 0 \\ 2\vec{q}^T & \vec{0}^T & 0 & 0 & 0 \\ 2\vec{p}^T & 2\vec{q}^T & 0 & 0 & 0 \end{pmatrix}. \tag{47}$$

As one readily checks, this matrix has only one zero mode $u^{(2)} = (\vec{0}^T, \vec{0}^T, 1, 0, 0)$ which, however, just reproduces the constraint $\varphi^{(0)} = 0$. Hence the algorithm terminates at this point, leaving us with a non-invertible matrix[3]. On the other hand, one readily checks that the standard Lagrangian (or equivalently, Dirac) algorithm generates not only the constraints $\vec{q}^2 - 1 = 0, \vec{p} \cdot \vec{q} = 0$, but also one further constraint $2\lambda\vec{q}^2 + \vec{p}^2 = 0$. Indeed, in the Lagrangian algorithm, $F^{(2)}$ in (47) is replaced by the *rectangular* matrix

$$W^{(2)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \\ 2\vec{q}^T & \vec{0}^T & 0 \\ 2\vec{p}^T & 2\vec{q}^T & 0 \end{pmatrix} \tag{48}$$

which is seen to possess the two level one zero modes enhanced by an additional zero entry, $u^{(2)}(1) = (\vec{0}, \vec{0}, 1, 0, 0)$, $u^{(2)}(2) = (\vec{0}, 2\vec{q}, 0, -1, 0)$, which just reproduce the previous constraints $\varphi^{(0)} = 0, \varphi^{(1)} = 0$, as well as the new zero mode

$$u^{(2)}(3) = (2\vec{q}, -2\vec{p}, 0, 0, 1) \tag{49}$$

[3] In [5] the Lagrange multiplier λ was absorbed into the dynamical variable $\rho^{(0)}$. Thereby the information about λ was lost, and the resulting matrix $F^{(2)}$ at level 2 became invertible.

which implies a new constraint

$$\varphi^{(2)} := 2\lambda\vec{q}^2 + \vec{p}^2 = 0. \tag{50}$$

Hence we are taken to a third level with the corresponding enlarged *rectangular* matrix given by

$$W^{(3)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \\ 2\vec{q}^T & \vec{0}^T & 0 \\ 2\vec{p}^T & 2\vec{q}^T & 0 \\ 4\lambda\vec{q}^T & 2\vec{p}^T & 2\vec{q}^2 \end{pmatrix}. \tag{51}$$

As one readily checks, $W^{(3)}$ has no *new* zero modes. Hence the algorithm terminates at this point. Note that the extension of this matrix to a square matrix as discussed in section 2 results in an invertible matrix, reflecting a second-class system.

We see that the symplectic algorithm fails to generate the correct set of constraints known to be present for the model in question. In fact, from the point of view of the *second-order* Lagrangian formulation there exists just one primary constraint $\phi = p_\lambda = 0$, where p_λ is the momentum conjugate to the variable λ, and the total Hamiltonian correspondingly reads, $H_T = \frac{1}{2}\vec{p}^2 - \lambda(\vec{q}^2 - 1) + vp_\lambda$. As one readily checks, the last constraint (50) just serves to fix the Lagrange multiplier v in H_T to $v = 0$. Only at this final stage does the second-class nature of the model in question become evident. If we stop at level two, v remains arbitrary, as expressed by the zero column in (47) and (48).

3.2. *When does the symplectic algorithm fail?*

We now examine in general terms at which point the symplectic algorithm begins to fail. To this end, we examine what the symplectic algorithm described above corresponds to on a Hamiltonian level. Let $L^{(0)}$ be of the form (1), with (27) the corresponding primary constraints. At the $(\ell + 1)$th level, the symplectic algorithm leads to a Lagrangian of the form (we streamline the notation)

$$L^{(\ell+1)} = L^{(0)} - \sum_{a_\ell} \dot{\rho}_{a_\ell}\varphi_{a_\ell}(Q)$$

where $\varphi_{a_\ell}, a_\ell = 1, \ldots, n_\ell$ denote *all* the constraints generated by the iterative procedure up to level ℓ. The corresponding total Hamiltonian reads

$$H_T^{(\ell+1)} = H_T^{(0)} + \sum_{a_\ell} \lambda_{a_\ell}\phi_{a_\ell}.$$

Here $\{\phi_{a_\ell}\}$ denote the corresponding set of primary constraints associated with $\{\dot{\rho}_{a_\ell}\}$,

$$\phi_{a_\ell} = P_{a_\ell} + \varphi_{a_\ell}(Q)$$

where P_{a_ℓ} are the momenta conjugate to the dynamical variables ρ_{a_ℓ} and

$$H_T^{(0)} = H^{(0)} + \sum_\alpha v_\alpha\phi_\alpha$$

with ϕ_α the primary constraints (27), associated with the original Lagrangian $L^{(0)}$. Hence in the symplectic algorithm described above, the total Hamiltonian is modified at each level. Clearly, the Euler–Lagrange equations derived from $L^{(\ell+1)}$ and the Hamilton equations of motion following from $H^{(\ell+1)}$ describe the same dynamics.

Conservation of all the primary constraints requires

$$\begin{aligned} \left\{\phi_\alpha, H_T^{(\ell+1)}\right\} &= -\frac{\partial V}{\partial Q_\alpha} + \sum_\beta \{\phi_\alpha, \phi_\beta\} v_\beta + \sum_{b_\ell} \left\{\phi_\alpha, \phi_{b_\ell}\right\} \lambda_{b_\ell} = 0 \\ \left\{\phi_{a_\ell}, H_T^{(\ell+1)}\right\} &= \sum_\beta \left\{\phi_{a_\ell}, \phi_\beta\right\} v_\beta + \sum_{b_\ell} \left\{\phi_{a_\ell}, \phi_{b_\ell}\right\} \lambda_{b_\ell} = 0. \end{aligned} \tag{52}$$

Let Φ_{A_ℓ} and λ_{A_ℓ} stand for

$$\Phi_{A_\ell} := \left(\phi_\alpha, \phi_{a_\ell}\right) \qquad \lambda_{A_\ell} := \left(v_\alpha, \lambda_{a_\ell}\right).$$

Then we may write (52) in the compact form

$$\sum_{B_\ell} \left\{\Omega_{A_\ell}, \Omega_{B_\ell}\right\} \lambda_{B_\ell} = K_{A_\ell} \tag{53}$$

where $\vec{K} = (\vec{\nabla} V, \vec{0})$, with $\vec{0}$ an $(N_\ell = n + n_\ell)$-component null vector.

One readily checks that $\left\{\Omega_{A_\ell}, \Omega_{B_\ell}\right\}$ is identical with $F_{A_\ell B_\ell}$ in (14) at the ℓth level. Furthermore, with the identification of v_α and λ_{a_ℓ} with $\dot{Q}_\alpha$ and $\dot{\rho}_{a_\ell}$ via the Hamilton equations of motion,

$$\dot{Q}_\alpha = \left\{Q_\alpha, H_T^{(\ell+1)}\right\} = v_\alpha$$

$$\dot{\rho}_{a_\ell} = \left\{\rho_{a_\ell}, H_T^{(\ell+1)}\right\} = \lambda_{a_\ell}$$

we see that the *persistence* equations (52) are just the equations of motion obtained from $L^{(\ell+1)}$ in the symplectic approach.

Within the Hamiltonian formalism, the search for zero modes of F at level ℓ now corresponds to seeking linear combinations of *all* the primaries, $u_{A_\ell} \Phi_{A_\ell}$, such that

$$\sum_{A_\ell} u_{A_\ell} \left\{\Phi_{A_\ell}, \Phi_{B_\ell}\right\} = 0. \tag{54}$$

From (53) we see that these equations imply linearly independent (non-trivial) constraints, which we denote by

$$\varphi_{a_\ell} = \sum u_{A_\ell}(a_\ell) K_{A_\ell} = 0.$$

Of the conditions (54), only those with $B_\ell = \beta$,

$$\sum_{A_\ell} u_{A_\ell}(a_\ell) \left\{\Phi_{A_\ell}, \phi_\beta\right\} = 0 \tag{55}$$

are contained in the Lagrangian (and hence traditional Dirac) approach. Let $u(a_\ell)$ be solutions of (55). From (54), with $B_\ell = b_\ell$ we see that the symplectic approach thus implies the *additional* restrictions

$$u_\alpha(a_\ell) \frac{\partial \varphi_{b_\ell}}{\partial Q_\alpha} = 0 \tag{56}$$

for the zero modes, which are *not* contained in the Lagrangian algorithm. Hence we have a mismatch between the symplectic and Lagrangian algorithms, once the latter condition is not satisfied in the iterative process, and the constraint structure becomes inequivalent for the two algorithms. This is the main point of this paper[4].

Let us exemplify this for the case of the 'particle on a hypersphere'. At the second level the Lagrangian algorithm leads to the new zero mode (49). We verify that at the zeroth and

[4] This subtle point has been missed in [7].

first level of the iterative process the condition (56) is still verified, whereas this is not the case for the second-level zero mode $u^{(2)}(3)$ in (49), since

$$u_\alpha^{(2)}(3)\frac{\partial\varphi^{(0)}}{\partial Q_\alpha} = 4\vec{q}^2 \neq 0.$$

This explains why the algorithm stops before generating one further constraint, $\varphi^{(2)} = 0$, equation (50).

It is instructive to further elucidate the meaning of this finding. Going through the iterative procedure on the Hamiltonian level (found above to be equivalent to the symplectic algorithm), we arrive after the second iterative step at the Hamiltonian

$$H_T^{(2)} = V^{(0)}(q,p,\lambda) + \sum_{i=1}^{n}\left(v_{q_i}\left(P_{q_i} - p_i\right) + v_{p_i}P_{p_i}\right) + vp_\lambda + \lambda_1(P_1 + \vec{q}^2 - 1) + \lambda_2(P_2 + 2\vec{q}\cdot\vec{p}).$$

Conservation in time of the primaries now merely serves to fix all the Lagrange multipliers $\lambda, \lambda_1, \lambda_2, \lambda_2$, and leads to:

$$\vec{v}_p + 2\lambda_1\vec{q} + 2\lambda_2\vec{p} = 2\lambda\vec{q} \qquad \vec{v}_q + 2\lambda_2\vec{q} = \vec{p} \qquad \vec{v}_q\cdot\vec{p} + \vec{v}_p\cdot\vec{q} = 0 \qquad \vec{v}_q\cdot\vec{q} = 0$$

as well as the constraint $\vec{q}^2 - 1 = 0$. These equations may be solved for λ_1 and λ_2,

$$\lambda_1 = \tfrac{1}{2}(\vec{p}^2 + 2\lambda - 2\vec{p}\cdot\vec{q}) \qquad \lambda_2 = \tfrac{1}{2}(\vec{p}\cdot\vec{q}) \tag{57}$$

and hence for v_{q_i} and v_{p_i}, leaving v undetermined. Hence $\vec{q}^2 - 1 = 0$ is the only constraint (as reflected by the zero column in (47) and (48)), unless we set $\lambda_1 = \lambda_2 = 0$. This just corresponds to working with the total Hamiltonian $H_T^{(0)}$. In that case the algorithm does not terminate, but rather generates one further constraint, $\vec{p}^2 + 2\lambda = 0$, whose time independence will finally fix also v to vanish.

4. Conclusion

In this paper, we have examined the interrelation between three different algorithms currently in use for unravelling the constrained structure of first-order Lagrangians. We have referred to these as the 'Lagrangian', 'Dirac' and 'BW symplectic' algorithms. Of these the first two rest on a solid foundation, and, as we have seen, there exists a one-to-one correspondence between these formalisms. In particular, we have shown how to invert the seemingly overdetermined system of equations of the Lagrangian algorithm. As for the symplectic algorithm presented in [5], it does not always reproduce the correct set of constraints, as we have seen. In fact, we have shown for a general first-order Lagrangian, under what conditions the algorithm fails to reproduce all of the constraints correctly. A concrete example has exemplified this.

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